ON THE ELASTIC POTENTIAL OF CRYSTALS.

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The theory of elasticity in its applications to homogeneous isotropic bodies has been more or less completely developed by Poisson, Cauchy, Lamé and Clebsch, while Riemann's "Differential-gleichungen" contains an admirable abstract of it.

On a particle of a solid body may act three kinds of forces:

1) *External forces*, attractions and repulsions by external masses;

2) *Surface-forces*, pressures or tensions applied directly to elements of the surface;

3) *Elastic forces*, molecular forces due to the influence of neighboring particles of the body itself.

Elastic forces are produced by unequal displacements of the particles, causing a change in their relative positions, and hence a change in the molecular forces acting between them. Surface-forces affect the internal particles of the body only indirectly, by causing a displacement of the particles of the surface, giving rise to elastic forces which spread through the interior. It is the elastic forces alone that I shall consider in this paper. These may be considered as forces tending to prevent any change in the relative positions of the particles on one side of a section (plane or curved) of the body with respect to those on the other side; and, because the mutual action of the particles becomes inappreciable when the distance between them exceeds a very small limit, a particle on one side of the section will be affected only by those particles on the other side which lie very near the section and within a small distance of the particle in question. So that the whole mutual effect of the particles on the two sides of the section may be treated as the sum of the mutual effects of particles on the two sides of the very small portions (which may be considered plane-elements) of the section; and it is only necessary to consider the elastic forces on such plane-elements, which vary from element to element of the section, but, in the immediate vicinity of any given point of the section, are proportional to the plane-elements on which they act. It is there-

fore convenient to call the *components* of the elastic force acting on any plane-element, the ratios of the actual components to the area of the element. The components of the elastic force acting on any plane-element passing through a given point may be expressed in terms of the components of the elastic forces acting on any three mutually perpendicular plane-elements through the point, *e. g.* three elements parallel to the planes of a rectangular system of coordinates.

Let x, y, z be the coordinates of any point of the body referred to a fixed rectangular system, and let three plane-elements of areas ε_x, ε_y, ε_z, whose normals are parallel to the axes of x, y, z respectively, pass through this point; then in notation of Cauchy the components of the elastic forces acting on ε_x, ε_y, ε_z in the directions of the three axes will be X_x, Y_x, Z_x; X_y, Y_y, Z_y; X_z, Y_z, Z_z: in each of which the large letter denotes the direction of the component, and the suffix the direction of the normal to the plane-element on which it acts. Let, further, the density at the point x, y, z be ρ.

If x, y, z is a point in the interior of the body, let the components of the resultant of the external forces acting on a very small mass m at this point be mX, mY, mZ; then the conditions for the equilibrium of this small mass are

$$(1)\quad \left\{\begin{aligned} &\rho X + \frac{\delta X_x}{\delta x} + \frac{\delta X_y}{\delta y} + \frac{\delta X_z}{\delta z} = 0, \\ &\rho Y + \frac{\delta Y_x}{\delta x} + \frac{\delta Y_y}{\delta y} + \frac{\delta Y_z}{\delta z} = 0, \\ &\rho Z + \frac{\delta Z_x}{\delta x} + \frac{\delta Z_y}{\delta y} + \frac{\delta Z_z}{\delta z} = 0, \\ &Y_z = Z_y\,,\ Z_x = X_z\,,\ X_y = Y_x\,. \end{aligned}\right.$$

If, however, x, y, z is a point of the surface, let α, β, γ be the direction-cosines of the inner normal to the surface, and $\varepsilon\Xi$, εH, εZ the components of the surface-force on an element ε of the surface; then the conditions for the equilibrium of an infinitesimal solid element immediately under the surface-element ε are

$$(2)\quad \left\{\begin{aligned} &\Xi + \alpha X_x + \beta X_y + \gamma X_z = 0, \\ &H + \alpha Y_x + \beta Y_y + \gamma Y_z = 0, \\ &Z + \alpha Z_x + \beta Z_y + \gamma Z_z = 0, \end{aligned}\right.$$

where the surface-force is considered positive if a pressure and negative if a tension.

It is well known that, if u, v, w are the displacements, in the directions of the axes, of the point x, y, z, and if

$$(3) \quad \left\{ \begin{aligned} & \frac{\delta u}{\delta x} = x_x \,, \; \frac{\delta v}{\delta y} = y_y \,, \; \frac{\delta w}{\delta z} = z_z \,, \\ & \frac{\delta v}{\delta z} + \frac{\delta w}{\delta y} = y_z = z_y \,, \; \frac{\delta w}{\delta x} + \frac{\delta u}{\delta z} = z_x = x_z \,, \; \frac{\delta u}{\delta y} + \frac{\delta v}{\delta x} = x_y = y_x \,, \end{aligned} \right.$$

then there exists a function Φ of x_x, y_y, z_z, y_z, z_x, x_y, such that

$$(4) \quad \left\{ \begin{aligned} & X_x = \frac{\delta\Phi}{\delta x_x} \,, \; Y_y = \frac{\delta\Phi}{\delta y_y} \,, \; Z_z = \frac{\delta\Phi}{\delta z_z} \,, \\ & Y_z = Z_y = \frac{\delta\Phi}{\delta y_z} \,, \; Z_x = X_z = \frac{\delta\Phi}{\delta z_x} \,, \; X_y = Y_x = \frac{\delta\Phi}{\delta x_y} \,. \end{aligned} \right.$$

This function Φ, which may be called the *elastic potential,* is homogeneous of the second degree in x_x, y_y, z_z, y_z, z_x, x_y, and contains therefore in general 21 terms, whose coefficients are characteristic for the body under consideration, depending only on its structure in different directions. For a homogeneous body, *i. e.* one, every particle of which is similarly surrounded by particles, the coefficients of Φ are constant, but for a non-homogeneous body they differ from point to point, *i. e,* are functions of the coordinates x, y, z. For a homogeneous isotropic body the form of Φ has been determined to be

$$(5) \qquad \Phi = \tfrac{1}{2}\lambda\,(x_x + y_y + z_z)^2 + \mu\,(x_x^2 + y_y^2 + z_z^2) + \tfrac{1}{2}\,\mu\,(y_z^2 + z_x^2 + x_y^2)\,,$$

where λ and μ are constants. I now propose to determine its form for homoneous crystalline bodies.

Each crystalline system has a form of symmetry with respect to three mutually perpendicular axes which is not altered by a transformation to any one of certain other systems of mutually perpendicular axes having the same origin. We may then take one of these systems of axes as the basis of a system of coordinates x, y, z, and any other of them as the basis of a system of coordinates x', y', z'. The external form of symmetry of a crystal must, it would seem, be also a form of symmetry of arrangement of its particles; if this be so, a transformation of coordinates from x, y, z to x', y', z' must leave the coefficients unchanged. This will be the basis of the following determinations of the relations between the coefficients, on which relations the form of Φ depends.

The transformation from x, y, z to x', y', z' can be effected by a combination of two simple kinds of transformations, viz. by inversions of axes and

rotations about axes through certain angles. There are six such transformations, which I will designate by the first six letters of the alphabet, as follows:

a) inversion of the axis of x,

$$x' = -x\,,\; y' = y\,,\; z' = z\,,\; u = -u'\,,\; v = v'\,,\; w = w'\,,$$
$$x_x = x'_{x'}\,,\; y_y = y'_{y'}\,,\; z_z = z'_{z'}\,,\; y_z = y'_{z'}\,,\; z_x = -z'_{x'}\,,\; x_y = -x'_{y'}\,;$$

b) inversion of the axis of y,

$$x' = x\,,\; y' = -y\,,\; z' = z\,,\; u = u'\,,\; v = -v'\,,\; w = w'\,,$$
$$x = x'_{x'}\,,\; y_y = y'_{y'}\,,\; z_z = z'_{z'}\,,\; y_z = -y'_{z'}\,,\; z_x = z'_{x'}\,,\; x_y = -x'_{y'}\,;$$

c) inversion of the axis of z,

$$x' = x\,,\; y' = y\,,\; z' = -z\,,\; u = u'\,,\; v = v'\,,\; w = -w'\,,$$
$$x_x = x'_{x'}\,,\; y_y = y'_{y'}\,,\; z_z = z'_{z'}\,,\; y_z = -y'_{z'}\,,\; z_x = -z'_{x'}\,,\; x_y = x'_{y'}\,;$$

d) rotation about the axis of x through the angle α,

$$x' = x\,,\; y' = y\cos\alpha + z\sin\alpha\,,\; z' = -y\sin\alpha + z\cos\alpha\,,$$
$$u = u'\,,\; v = v'\cos\alpha - w'\sin\alpha\,,\; w = v'\sin\alpha + w'\cos\alpha\,,$$
$$x_x = x'_{x'}\,,\; y_y = y'_{y'}\cos^2\alpha - y'_{z'}\sin\alpha\cos\alpha + z'_{z'}\sin^2\alpha\,,$$
$$z_z = y'_{y'}\sin^2\alpha + y'_{z'}\sin\alpha\cos\alpha + z'_{z'}\cos^2\alpha\,,$$
$$y_z = (y'_{y'} - z'_{z'})\sin 2\alpha + y'_{z'}\cos 2\alpha\,,$$
$$z_x = x'_{y'}\sin\alpha + z'_{x'}\cos\alpha\,,\; x_y = x'_{y'}\cos\alpha - z'_{x'}\sin\alpha\,;$$

e) rotation about the axis of y through the angle β, by formulæ found from those of *d*) by changing x to y, y to z, z to x, and α to β;

f) rotation about the axis of z through the angle γ, by formulæ found from *d*) by changing x to z, y to x, z to y, and α to γ.

The general form of Φ is

$$(6)\quad \left\{\begin{aligned} \Phi = a_{11} + x_x^2 + a_{22}y_y^2 + a_{33}z_z^2 + 2a_{23}y_yz_z + 2a_{13}x_xz_z + 2a_{12}x_xy_y \\ + 2a_{14}x_xy_z + 2a_{15}x_xz_x + 2a_{16}x_xx_y + 2a_{24}y_yy_z + 2a_{25}y_yz_x \\ + 2a_{26}y_yx_y + 2a_{34}z_zy_z + 2a_{35}z_zz_x + 2a_{36}z_zx_y + a_{44}y_z^2 \\ + a_{55}z_x^2 + a_{66}x_y^2 + 2a_{56}z_xx_y + 2a_{46}x_yy_z + 2a_{45}y_zz_x\,. \end{aligned}\right.$$

That the transformation *a*) shall leave Φ unaltered in form it is necessary and sufficient that

$$(7)\qquad a_{15} = a_{25} = a_{35} = a_{45} = a_{56} = a_{16} = a_{26} = a_{36} = a_{46} = 0.$$

That the transformation *b*) shall leave the form of Φ unaltered it is necessary and sufficient that

$$(8)\qquad a_{14} = a_{24} = a_{34} = a_{45} = a_{16} = a_{26} = a_{36} = a_{56} = 0.$$

That the transformation *c*) shall leave the form of Φ unaltered it is necessary and sufficient that

$$(9)\qquad a_{14} = a_{24} = a_{34} = a_{46} = a_{15} = a_{25} = a_{35} = a_{56} = 0.$$

That the transformation *d*) shall leave the form of Φ unaltered it is necessary and sufficient that

$$(10)\left\{\begin{array}{l}
(a_{12}-a_{13})\sin\alpha=0\,,\ (a_{55}-a_{66})\sin\alpha=0\,,\ (a_{22}-a_{33})\sin\alpha=0\,,\\
a_{14}\sin\alpha=0\,,\ a_{56}\sin\alpha=0\,,\ a_{15}(1-\cos\alpha)=0\,,\ a_{16}(1-\cos\alpha)=0\,,\\
(a_{22}+a_{33}-2a_{23}-4a_{44})\sin\alpha\cos\alpha=0\,,\\
(a_{24}+a_{34})\sin\alpha=0\,,\ (a_{24}-a_{34})\sin\alpha\cos\alpha=0\,,\\
(a_{25}+a_{35})(1-\cos\alpha)=0\,,\ (a_{26}+a_{36})(1-\cos\alpha)=0\,,\\
[(a_{25}-a_{35})+(a_{26}-a_{36})]\sin\alpha\cos\alpha+(\cos\alpha-\cos 2\alpha-\sin\alpha)\,a_{45}\\
+(\cos\alpha-\cos 2\alpha+\sin\alpha)\,a_{46}=0\,,\\
[(a_{25}-a_{35})-(a_{26}-a_{36})]\sin\alpha\cos\alpha+(\cos\alpha-\cos 2\alpha+\sin\alpha)\,a_{45}\\
-(\cos\alpha-\cos 2\alpha-\sin\alpha)\,a_{46}=0\,,\\
(a_{25}-a_{35})(\cos\alpha-\cos 2\alpha-\sin\alpha)+(a_{26}-a_{36})(\cos\alpha-\cos 2\alpha+\sin\alpha)\\
-4\,(a_{45}+a_{46})\sin\alpha\cos\alpha=0\,,\\
(a_{25}-a_{35})(\cos\alpha-\cos 2\alpha+\sin\alpha)-(a_{26}-a_{36})(\cos\alpha-\cos 2\alpha-\sin\alpha)\\
-4\,(a_{45}-a_{46})\sin\alpha\cos\alpha=0\,.
\end{array}\right.$$

The conditions (10) are, of course, satisfied by $\alpha=0$. They will also be satisfied by a multiple of 90° when certain relations hold between the coefficients, as follows:

if $\alpha=90°$ or $-90°$,

$$(11)\left\{\begin{array}{l}
a_{14}=a_{15}=a_{16}=a_{25}=a_{26}=a_{35}=a_{36}=a_{45}=a_{46}=a_{56}=0\,,\\
a_{12}=a_{13}\,,\ a_{22}=a_{33}\,,\ a_{55}=a_{66}\,,\ a_{24}=-a_{34}\,;
\end{array}\right.$$

if $\alpha=180°$,

$$(12)\qquad a_{15}=a_{25}=a_{35}=a_{45}=a_{16}=a_{26}=a_{36}=a_{46}=0\,,$$

which are identical with (7), so that, if an axis can be inverted without producing any effect on Φ, a rotation about that axis through an angle of 180° will also produce no effect, and *vice versa*, as is also evident from the fact that all three axes may be inverted without producing any effect on the most general form (6) of Φ.

If α has neither of the above values,

$$(13)\left\{\begin{array}{l}
a_{14}=a_{24}=a_{34}=a_{15}=a_{16}=a_{56}=0\,,\\
a_{12}=a_{13}\,,\ a_{22}=a_{33}=a_{23}+2a_{44}\,,\ a_{55}=a_{66}\,,\ a_{25}=-a_{35}\,,\ a_{26}=-a_{36}\,,
\end{array}\right.$$

and either

$$(13_1)\qquad a_{25}=a_{35}=a_{45}=a_{26}=a_{36}=a_{46}=0\,,$$

or

$$(13_2)\quad 4\cos^6\alpha+4\cos^5\alpha-2\cos^4\alpha-2\cos^3\alpha-4\cos^2\alpha-4\cos\alpha-1=0$$

(whose only real roots between -1 and $+1$ are $\cos\alpha = -0.607+$ and $\cos\alpha = -0.389-$), *i e.* if a rotation about an axis through any other angle than 90°, 180°, 270° (or $-90°$), $\cos^{[-1]}(-0.607+)$ or $\cos^{[-1]}(-0.389-)$ does not affect the form of Φ, a rotation about this axis through any angle whatever will not affect it, *i. e.* there exists perfect symmetry about this angle.

The transformations *e*) and *f*) give conditions similar to (10), (11), (12) and (13), found from them by a cyclic interchange in each set of suffixes 1, 2, 3 and 4, 5, 6 and in α, β, γ.

Each of the crystalline systems admits of a characteristic combination of the above transformations.

The *triclinic* system admits only of an inversion of all the axes at the same time, which leaves the most general form of Φ unchanged without any conditions between the coefficients, *i. e.* (6) is the general form of Φ for triclinic crystals.

The *monoclinic* system, in which the axis of x is taken in that axis of the crystal which is perpendicular to the plane of the two others, allows only the transformation *a*) and *d*) in which $\alpha = 180°$, *i. e.* its conditions are (7) and (12), which are identical, and

$$(14)\quad \left\{\begin{aligned} \Phi = {} & a_{11}x_x^2 + a_{22}y_y^2 + a_{33}z_z^2 + 2a_{23}y_yz_z + 2a_{13}x_xz_z \\ & + 2a_{12}x_xy_y + 2a_{14}x_xy_z + 2a_{24}y_yy_z + 2a_{34}z_zy_z + a_{44}y_z^2 \\ & + a_{55}z_x^2 + a_{66}x_y^2 + 2a_{56}z_xx_y\,. \end{aligned}\right.$$

The *trimetric* system, in which the axes of x, y and z coincide with the axes of the crystal, admits of each of the transformations *a*), *b*) and *c*), hence for this system the conditions are (7), (8) and (9), and the form of Φ is

$$(15)\quad \left\{\begin{aligned} \Phi & = a_{11}x_x^2 + a_{22}y_y^2 + a_{33}z_z^2 + 2a_{23}y_yz_z + 2a_{13}x_xz_z + 2a_{12}x_xy_y \\ & = a_{44}y_z^2 + a_{55}z_x^2 + a_{66}x_y^2. \end{aligned}\right.$$

The *dimetric* system, whose principal axis is the axis of x and whose secondary axes are those of y and z, admits of the transformations *a*), *b*), *c*) and *d*) where $\alpha = 90°$, so that the conditions are (7), (8), (9) and (11), and the form of Φ is

$$(16)\quad \left\{\begin{aligned} \Phi = {} & a_{11}x_x^2 + a_{22}\,(y_y^2 + z_z^2) + 2a_{23}y_yz_z + 2a_{12}x_x\,(y_y + z_z) \\ & + a_{44}y_z^2 + a_{55}\,(z_x^2 + x_y^2). \end{aligned}\right.$$

The *monometric* system, whose axes are the axes of x, y and z, admits of the transformations *a*), *b*), *c*), *d*), *e*) and *f*) for $\alpha = 90°$, $\beta = 90°$, $\gamma = 90°$; so that the conditions are (7), (8), (9), (11) and, similar to (11),

$$a_{14} = a_{16} = a_{24} = a_{25} = a_{26} = a_{34} = a_{36} = a_{56} = a_{46} = a_{45} = 0\,,$$
$$a_{23} = a_{12}\,,\ a_{11} = a_{33}\,,\ a_{44} = a_{66}\,,\ a_{15} = -\,a_{35}\,,$$
$$a_{14} = a_{15} = a_{24} = a_{25} = a_{34} = a_{35} = a_{36} = a_{56} = a_{46} = a_{45} = 0\,,$$
$$a_{23} = a_{13}\,,\ a_{11} = a_{22}\,,\ a_{44} = a_{55}\,,\ a_{16} = -\,a_{26}\,,$$

and the form of Φ is

$$(17)\quad \Phi = a_{11}\,(x_x^3 + y_y^2 + z_z^2) + 2a_{12}\,(y_y z_z + z_z x_x + x_x y_y) + a_{44}\,(y_z^2 + z_x^2 + x_y^2).$$

The *hexagonal*, as well as *any regular prismatic* system, other than one whose secondary axes include an angle of 90° (because the conditions (13_1) omitted in case $\alpha = \cos^{[-1]}(-0.607+)$ or $\alpha = \cos^{[-1]}(-0.389-)$ are more than replaced by (7)), whose principal axis is that of x, admits of the transformations *a*), *b*), *c*), *d*); the conditions are then (7), (8), (9), (13) and (13_1), and Φ has the form

$$(18)\quad \left\{ \begin{aligned} \Phi = a_{11}x_x^2 + a_{22}\,(y_y^2 + z_z^2) + 2a_{23}y_y z_z + 2a_{12}x_x\,(y_y + z_z) \\ + \tfrac{1}{2}\,(a_{22} - a_{23})\,y_z^2 + a_{55}\,(z_x^2 + x_y^2)\,. \end{aligned} \right.$$

It is to be remarked that (16) differs from (18) only in the coefficient of y_z^2, which is independent of the other coefficients in the former case, but dependent on the coefficients of $y_y^2 + z_z^2$ and $y_y\,z_z$ in the latter case, so that, as far as their elastic relations are concerned, square prismatic crystals seem to be the most complex of regular prismatic forms, in as much as they contain one more constant than any other.

From the above determinations of Φ it will be seen that the elastic potentials of isotropic bodies, of monometric, hexagonal, dimetric, trimetric, monoclinic and triclinic crystals contain respectively 2, 3, 5, 6, 9, 13 and 21 constants. The forms of Φ, (5), (17), (18), (16), (15,) (14) and (6) substituted in (4) give the forms of the components of the elastic force in these different cases of homogeneity, and these components substituted in (1) and (2) give the conditions for equilibrium under a given set of external forces X, Y, Z and surface-forces Ξ, H, Z. The conditions for motion, *e. g.* elastic vibrations, will be found from those for equilibrium by the substitution of $X - \frac{d^2u}{dt^2}$, $Y - \frac{d^2v}{dt^2}$, $Z - \frac{d^2w}{dt^2}$ for X, Y, Z respectively, t denoting time.